

Against relative overlap measures of coherence

Jakob Koscholke¹ · Michael Schippers¹

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Abstract Coherence is the property of propositions hanging or fitting together. Intuitively, adding a proposition to a set of propositions should be compatible with either increasing or decreasing the set’s degree of coherence. In this paper we show that probabilistic coherence measures based on relative overlap are in conflict with this intuitive verdict. More precisely, we prove that (i) according to the naive overlap measure it is impossible to increase a set’s degree of coherence by adding propositions and that (ii) according to the refined overlap measure no set’s degree of coherence exceeds the degree of coherence of its maximally coherent subset. We also show that this result carries over to all other subset-sensitive refinements of the naive overlap measure. As both results stand in sharp contrast to elementary coherence intuitions, we conclude that extant relative overlap measures of coherence are inadequate.

Keywords Bayesian coherentism · Probabilistic coherence measures · Relative overlap

1 Introduction

The concept of coherence plays the key role in any coherentist theory of epistemic justification or truth. Therefore, it is of vital importance for the tenability of such theories to render precise this concept. Striving for mathematical precision in their analyses of coherence, several authors have proposed so-called *probabilistic coherence*

✉ Jakob Koscholke
jakob.koscholke@uni-oldenburg.de

Michael Schippers
mi.schippers@uni-oldenburg.de

¹ Philosophy Department, University of Oldenburg, 26111 Oldenburg, Germany

measures. These measures typically presuppose the following formal framework: let L be a propositional language, i.e. a set of formulae closed under some functionally complete selection of classical connectives, e.g. $\{\neg, \wedge\}$ and let $P : L \rightarrow [0, 1]$ be a probability function over L , i.e. a non-negative, real-valued function where $P(x) = 1$ if $x \in L$ is a tautology and $P(x_1 \vee x_2) = P(x_1) + P(x_2)$ if $x_1, x_2 \in L$ are logically incompatible. Furthermore, let $2_{\geq 2}^L$ denote the set of all subsets of L with at least two propositions¹ and \mathbf{P} be the set of all regular probability functions over L , i.e. probability functions where $P(x) = 1$ if and only if $x \in L$ is a tautology and $P(x) = 0$ if and only if x is a contradiction (cf. Shimony 1955). Then, a probabilistic measure of coherence is a (partial) function $C : 2_{\geq 2}^L \times \mathbf{P} \rightarrow \mathbb{R}$ assigning each pair (X, P) a real number that is supposed to represent X 's degree of coherence under P . Notice that, as usual in this context, we omit reference to the particular probability function as a separate function argument unless it is needed. Prominent proposals for probabilistic coherence measures can be pigeonholed into the following classes: *deviation from independence measures* (cf. Shogenji 1999; Schupbach 2011), *relative overlap measures* (cf. Glass 2002; Olsson 2002; Meijs 2006) and *mutual support measures* (cf. Fitelson 2003; Douven and Meijs 2007; Roche 2013; Schippers 2014a).²

In this paper we will focus on relative overlap measures of coherence as it can easily be seen that other probabilistic coherence measures are not affected by the points made in this paper. To be more specific, we will put forward two arguments against extant overlap measures that highlight the intuition that it should be possible to increase a set's degree of coherence by adding propositions. In Sect. 2 it is shown that this intuition is violated by the naive overlap measure of coherence. Section 3 then proceeds with a discussion of the refined overlap measure that is often considered an improvement of its naive counterpart. First of all, we show that this is true with respect to the former property of the naive measure, which is not shared by its refined counterpart. Nonetheless, we also show that the fact that the refined measure is, in a nutshell, only a weighted average of values of its naive counterpart, also compromises its performance in related contexts. More precisely, it can be shown that according to the refined overlap measure each set is at most as coherent as its maximally coherent subset (and usually lower). We argue that this is inadequate and therefore conclude that no extant overlap measure captures the concept of coherence. Finally, we show that this result does not depend on the fact that the refined overlap measure is based on the straight average of values of the naive overlap measure but carries over to all other subset-sensitive refinements of the naive measure.

¹ This restriction has been called "Rescher's principle" (Olsson 2005, p. 17) and basically amounts to the fact that coherence is a property that propositions cannot have in isolation but only in groups of at least two propositions (cf. Rescher 1973, p. 32). For exceptions to Rescher's principle in discussions on probabilistic coherence measures see Akiba's (2000) and Fitelson's (2003) discussions on self-coherence.

² Probabilistic measures of coherence have been discussed critically by Bovens and Hartmann (2003), Moretti and Akiba (2007), Olsson (2005), Olsson and Schubert (2007), Siebel (2005) and Siebel and Wolff (2008). For an overview of the measures and their structural properties see Schippers (2014a, b, 2015), for an overview of their performance in a collection of test cases see Koscholke (2015). The relative overlap measure has received special attention in the literature due to its high degree of truth-conduciveness as shown by Angere (2007, 2008) and its strong performance in inferences to the best explanation as presented by Glass (2012).

2 The naive overlap measure

In two independent articles [Glass \(2002\)](#) and [Olsson \(2002\)](#) have proposed the following mathematical function as a probabilistic measure of the degree of coherence of some set X under a probability distribution P :

$$\mathcal{O}(X) = \frac{P\left(\bigwedge_{x_i \in X} x_i\right)}{P\left(\bigvee_{x_i \in X} x_i\right)}$$

The numerator of the measure is the probability that all propositions in X are true together while the denominator is the probability that at least one proposition in X is true. Set-theoretically speaking the numerator can be understood as the absolute overlap of X 's propositions whereas the denominator can be understood as the total surface of X 's propositions. The basic idea of this measure is therefore often referred to as measuring coherence in terms of *relative set-theoretic overlap*. For reasons that will become clear later we will refer to Glass and Olsson's measure as the *naive* overlap measure. The measure's codomain is $[0, 1]$ so the measure takes its minimum value only if there is no absolute overlap but a non-empty total surface and its maximum only if the absolute overlap and the total surface are identical. Particularly, the measure assigns a minimum degree of coherence to inconsistent propositions as long as the denominator is not 0 and a maximum degree of coherence to logically equivalent propositions. It is also worth noticing that the Glass–Olsson measure is undefined if the probability of at least one proposition in a given set being true is equal to zero.

Though at first sight it might seem appealing as a probabilistic measure of coherence, [Bovens and Hartmann \(2003\)](#) have formulated a test case in which the measure yields counter-intuitive results. Suppose that we are provided with information from independent and equally reliable sources that someone's pet Tweety is a bird (x_1) and that Tweety is a ground-dweller (x_2). Later, we also receive the information that Tweety is a penguin (x_3). Bovens and Hartmann assume the following joint probability distributions over the set $\{x_1, x_2, x_3\}$ (Fig. 1).

Fig. 1 Probability distribution for the Tweety case

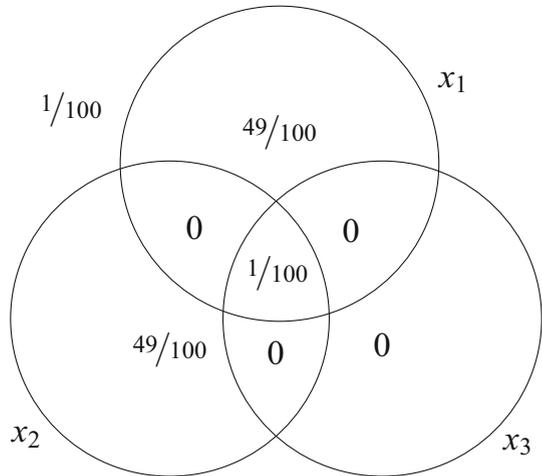


Table 1 Results in the Tweety case

Tweety sets	θ	θ'
$\{x_1, x_2\}$	0.010	0.010
$\{x_1, x_2, x_3\}$	0.010	0.015

Intuitively, so Bovens and Hartmann argue, the extended set $\{x_1, x_2, x_3\}$ is more coherent than the reduced set $\{x_1, x_2\}$ since given our background knowledge about penguins the information that Tweety is a penguin entails that it is a bird and that it is a ground-dweller and therefore x_3 relieves the tension between the two propositions x_1 and x_2 . Nevertheless, the naive measure violates this intuition. It can easily be seen in Table 1 that both sets are assigned identical degrees of coherence (cf. Bovens and Hartmann 2003, p. 50).

Now, this result is well-documented in the philosophical literature. However, what has gone unnoticed is that this test case only exemplifies a general problem for the naive overlap measure which is far more devastating. In its most general form, this problem amounts to the following observation:

Theorem 1 For any P over L and $X \in 2^L_{\geq 2}$: if $X' \subset X$, then $\theta(X') \geq \theta(X)$.

Thus, according to the naive overlap measure each subset of a given set of propositions is at least as coherent as the given set itself. To see this more clearly, recall that a joint probability function P over a set of propositions $X = \{x_1, \dots, x_n\}$ which might exhaust the underlying language L can be represented as follows:

x_1	x_2	...	x_{n-1}	x_n	P
0	0	...	0	0	p_1
0	0	...	0	1	p_2
0	0	...	1	0	p_3
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
1	1	...	0	1	p_{2^n-2}
1	1	...	1	0	p_{2^n-1}
1	1	...	1	1	p_{2^n}

In this representation, p_1 is the probability that all $x_i \in X$ are false together whereas p_{2^n} is the probability that all $x_i \in X$ are true together. The naive overlap measure can then be reformulated the following way:

$$\mathcal{O}(X) = \frac{p_{2^n}}{\sum_{i=2}^{2^n} p_i}$$

Similarly, let $X' = \{x_1, \dots, x_m\}$ for $m \leq n$, then we get analogously:

$$\mathcal{O}(X') = \frac{\sum_{i=2^{n-(2^{n-m}-1)}}^{2^n} p_i}{\sum_{j=2^{n-m}+1}^{2^n} p_j}$$

Given that p_{2^n} is at most as high as \mathcal{O} 's numerator for X' and that $\sum_{i=2}^{2^n} p_i$ is at least as high as \mathcal{O} 's denominator for X' , we immediately get the result that $\mathcal{O}(X) \leq \mathcal{O}(X')$, with equality holding if and only if $p_{2^{n-1}} = \dots = p_{2^{n-(2^{n-m}-1)}} = 0$ and $p_{2^{n-m}} = \dots = p_{2^1} = 0$. Otherwise, X is *less* coherent according to \mathcal{O} .

This general result aggravates the former test case finding in the following way: even if one were inclined to judge the particular result in the test case as an artifact of the chosen probability distribution or a negligible exception for a measure that otherwise performs well, Theorem 1 can be considered a definite knock-down argument for the naive overlap measure of coherence. A coherence measure based on relative overlap seems to be of little or no epistemological value if it does not allow for the degree of coherence of a set of propositions to increase if a proposition is added. However, there seems to be a loophole out of this awkward position: perhaps this result is solely due to the way the naive overlap measure is generalized to the case of n propositions. Accordingly, the following section dwells upon the refined counterpart of \mathcal{O} which has been proposed by Meijs (2006).

3 The refined overlap measure

The refined overlap measure is based on the insight that the degree of coherence of a given set of propositions is sensitive to the coherence of its subsets. Therefore, when measuring coherence according to the refined overlap measure, \mathcal{O} is not only applied to the given set of propositions, but also to all of its subset with at least two propositions. For a concise representation of the measure let $2_{\geq 2}^X$ denote the set of all subsets of X with at least two members and with cardinality $|2_{\geq 2}^X| = (2^n - n) - 1$. Then the refined overlap measure can be characterized as follows:

$$\mathcal{O}'(X) = \frac{\sum_{X' \in 2_{\geq 2}^X} \mathcal{O}(X')}{(2^n - n) - 1}$$

One of the success stories that gave reason to prefer \mathcal{O}' over \mathcal{O} is that it masters Bovens and Hartmann's Tweety case, i.e. the extended set $\{x_1, x_2, x_3\}$ is judged *more* coherent

than the reduced set $\{x_1, x_2\}$ (cf. Meijs 2006, p. 245) as can be seen in Table 1.³ This already entails that \mathcal{O}' does not fall prey to the above knock-down argument against the naive overlap measure \mathcal{O} .

However, there is an argument against \mathcal{O}' that is associated with the fact that \mathcal{O}' is only an average of \mathcal{O} -values and therefore inherits some of its problems. Especially, the property mentioned in Theorem 1, viz. that according to \mathcal{O} it is impossible to increase a set's coherence by adding propositions, gives rise to a related problem for Meijs' measure \mathcal{O}' . To illustrate the problem, consider BonJour's well-known raven example (cf. BonJour 1985, p. 96): suppose again that we are provided with information from independent and equally reliable sources that all ravens are black (x_1), that someone's pet Henry is a raven (x_2) and that Henry is black (x_3). The assumption that the set $\{x_1, x_2, x_3\}$ should be assigned a significant degree of coherence can be motivated as follows: "[...] the component propositions, rather than being irrelevant to each other, fit together or reinforce each other in a significant way; from an epistemic standpoint, any two of them would lend a degree of positive support to the third" (BonJour 1985, p. 96). Moreover, it seems obvious that all proper subsets of $\{x_1, x_2, x_3\}$ containing at least two propositions are intuitively less coherent than the set itself. It is the information that all ravens are black that really ties all propositions together. This intuition is almost reversed when we apply the refined overlap measure to the following probability distribution over $\{x_1, x_2, x_3\}$ that has been given by Bovens and Hartmann (2003) in their discussion of BonJour's raven case (Fig. 2).

The degrees of coherence of $\{x_1, x_2, x_3\}$ and its two-membered subsets are given in Table 2.

Obviously, both overlap measures disagree considerably with the intuitive verdict since both assign a *higher* degree of coherence to at least some subsets of $\{x_1, x_2, x_3\}$. In the light of Theorem 1, this should come as no surprise for the naive overlap measure \mathcal{O} . But this property of \mathcal{O} also affects its refined counterpart. The following theorem shows that the particular results in the raven case exemplify a general problem for \mathcal{O}' . For some set X let o_1, \dots, o_m denote the coherence values for each of the subsets X'_1, \dots, X'_m as assigned by \mathcal{O}' where o_m denotes the coherence value of the target and thus the largest set. Then the following relationship holds:

Theorem 2 For any P over L and $X \in \mathcal{L}_{\geq 2}^L$: $\mathcal{O}'(X) \leq \max(\{o_1, \dots, o_{m-1}\})$.

In other words, according to \mathcal{O}' , any set X is at most as coherent as its maximally coherent subset. To prove this observation note that $\mathcal{O}'(X') \geq \mathcal{O}'(X'')$ for each

³ Nonetheless, it is worth noticing that \mathcal{O}' has some flaws in the Tweety case. What if we had not received the information about Tweety being a penguin, i.e. x_3 , but rather $\neg x_3$, i.e. the information that Tweety is not a penguin? This proposition does not establish any inferential connections to x_1 or x_2 . Instead, it

Footnote 3 Continued

seems even to decrease coherence because given $\neg x_3$, one possible explanation for why x_1 and x_2 might be the case, our last resort of making sense of x_1 and x_2 in some sense, vanishes. Hence, the extended set $\{x_1, x_2, \neg x_3\}$ should be less coherent than $\{x_1, x_2\}$. Quite surprisingly, the Glass–Olsson naive overlap measure satisfies this intuition since $\mathcal{O}(\{x_1, x_2, \neg x_3\}) = 0$. Meijs' refined measure does not because $\mathcal{O}'(\{x_1, x_2, \neg x_3\}) = 0.247$. However, one should be careful with this modified case since it involves a negated proposition. Negations are known to be difficult to handle in intuitive judgements (cf. Deutsch et al. 2009).

Fig. 2 Probability distribution for the raven case

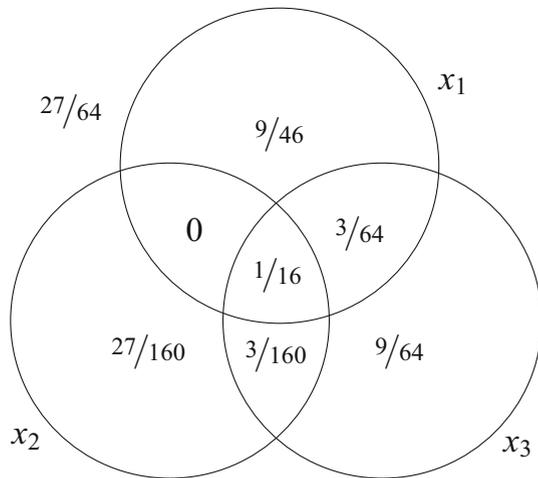


Table 2 Results in the raven case

Raven sets	\mathcal{O}	\mathcal{O}'
$\{x_1, x_2\}$	0.14	0.14
$\{x_1, x_3\}$	0.26	0.26
$\{x_2, x_3\}$	0.18	0.18
$\{x_1, x_2, x_3\}$	0.11	0.17

$X' \subset X'' \subset X$. This is due to the fact that the averaging procedure for $\mathcal{O}'(X')$ takes into account the coherence values of all subsets of $2^X_{\geq 2}$ that are also accounted for by $\mathcal{O}'(X'')$ and additionally some coherence values for subsets of X of higher cardinality. In the light of Theorem 1, these latter coherence values, however, cannot exceed the former. Accordingly, $\mathcal{O}'(X'')$ cannot exceed $\mathcal{O}'(X')$. This immediately entails that no subset X' can be assigned a higher \mathcal{O}' -coherence than any of the members of $2^X_{=2}$, i.e. the set of subsets of X with exactly two elements. In conclusion, $\mathcal{O}'(X)$ cannot exceed $\mathcal{O}'(X^*)$, where $X^* \subset X$ is the subset that is assigned the highest \mathcal{O}' -coherence and it is clear from the considerations above that candidates for X^* can only be sets with exactly two propositions.

In contrast to \mathcal{O}' , no other probabilistic coherence measure presented in the literature such as Shogenji's (1999), Fitelson's (2003), Douven and Meijs' (2007), Schupbach's (2011), Roche's (2013) or Schippers' (2014a) behaves like this. Simply consider the probability model given in Fig. 3 as a counterexample.

Based on some simple calculations one can easily see in Table 3 that according to the aforementioned measures (for their function equations the reader is kindly referred to the respective literature) each 2-element subset is less coherent than the complete 3-element set under this distribution of probabilities:

It should also be noted that the property affecting \mathcal{O}' does not depend on the fact that it uses the straight average over all \mathcal{O} values. Quite the contrary, it holds for any

Fig. 3 Probability distribution as counterexample

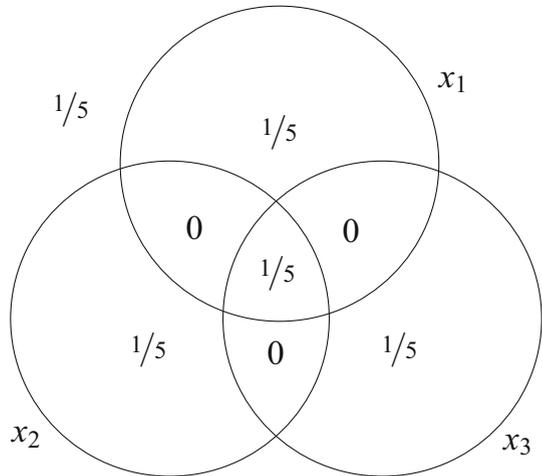


Table 3 Results for the distribution displayed in Fig. 3

Sets	Shogenji	Fitelson	Douven and Meijs	Schupbach	Roche	Schippers
$\{x_1, x_2\}$	1.25	0.2	0.1	0.096	0.500	0.25
$\{x_1, x_3\}$	1.25	0.2	0.1	0.096	0.500	0.25
$\{x_2, x_3\}$	1.25	0.2	0.1	0.096	0.500	0.25
$\{x_1, x_2, x_3\}$	3.125	0.5	0.275	0.295	0.625	0.5

averaging procedure. Accordingly, we introduce a recipe for refined overlap measures based on some average of \mathcal{O} -values. For each set X let X'_1, \dots, X'_m denote the elements of $2^X_{\geq 2}$. Then a weighted average of \mathcal{O} values can be defined by means of a weight vector $W = \langle w_1, \dots, w_m \rangle$ with $w_i \in [0, 1]$ such that $\sum_{i \leq m} w_i = 1$.

$$\mathcal{O}^W(X) = \sum_{i=1}^m w_i \cdot \mathcal{O}(X'_i)$$

This way it can easily be seen that Meijs’ version of the measure is just \mathcal{O}^W where $w_i = 1/m$ for any $w_i \in W$. Now, in order to prove Theorem 2 for each instance of \mathcal{O}^W observe that each $\mathcal{O}^W(X')$ for any $X' \subseteq X$ is, again, only some average of \mathcal{O} -values. Therefore, the above argument also entails that $\mathcal{O}^W(X')$ cannot exceed the degree of \mathcal{O}^W -coherence that is assigned to the maximally coherent two-membered subset of X for any W . A fortiori, \mathcal{O}^W cannot exceed the degree of \mathcal{O}^W -coherence of its maximally coherent subset.

This result, too, is devastating for the refined overlap measure of coherence. There seems to be no good reason why the fact described in Theorem 2 should hold for any set whatsoever. In particular, the raven test case illustrates that it should not hold for at least some sets. More general, any set $\{x_1 \rightarrow x_2, x_1, x_2\}$ under some probability function where each proposition has been provided by independent and equally reli-

able sources might be considered as a counterexample. Each subset $\{x_1 \rightarrow x_2, x_1\}$, $\{x_1 \rightarrow x_2, x_2\}$ and $\{x_1, x_2\}$ seems relatively unconnected.⁴ There is always one final piece of information missing to make each of the sets coherent. In the final set, however, all pieces of information fit together neatly. Therefore, it is hard to see why the value of a probabilistic measure of coherence should be limited by the degrees of coherence of the possibly unconnected subsets and not allow for a higher degree of coherence of the final set. Given the results presented above, we conclude that both kinds of relative overlap measures \mathcal{O} and \mathcal{O}^W , in particular \mathcal{O}' , cannot be adequate measures of coherence.

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⁴ Notice that the arguments of probabilistic coherence measures are usually not assumed to be closed under (classical) logical consequence.

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